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EXACT SOLUTIONS OF THE BURGERS–HUXLEY EQUATION†

O. Yu. YEFIMOVA and N. A. KUDRYASHOV

Moscow

e-mail: mephistka@mtu-net.ru; kudr@dampe.mephi.ru

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The Cole–Hopf transformation, that transforms the Burgers equation into the heat-conduction equation, is used to obtain exact solutions of the Burgers–Huxley equation, which occurs in the description of many non-linear wave phenomena. The types of exact solutions are analysed, depending on the values of the equation parameters.

1. INTRODUCTION

The Burgers–Huxley equation

$$U_t + \alpha U U_x = D U_{xx} + \beta U + \gamma U^2 - \delta U^3, \quad D \neq 0 \tag{1.1}$$

is encountered in the description of many non-linear wave phenomena. It is assumed that D is the diffusion coefficient, α characterizes non-linear transfer, and the parameters β, γ and δ describe a non-linear source.

For example, the equation

$$\tau P_t = l^2 P_{xx} + \varepsilon P - \varsigma P^3 - E \tag{1.2}$$

describing the motion of the domain wall of a ferroelectric material in an electric field, reduces, via a linear substitution, to Eq. (1.1) with $\alpha = 0$. In Eq. (1.2) P is dipole moment and E is the external electric field. The left-hand side characterizes the dissipative process in which electrostatic energy is converted into thermal energy, where τ is the characteristic duration of the process (that is, the relaxation time of the dipole moment). The first term on the right of the equation describes the interaction between the dipole moments of adjacent domains in the ferroelectric material; the other terms define the magnitude of the dipole moment in a homogeneous ferroelectric [1].

Equations like (1.1) are also used the description of certain ecological models. If a population is breeding in a medium, the dynamics of the system, taking into account mortality and the diffusion shift of the population through the medium, is described by the equation

$$n_t = -kn + \kappa m(n) n^2 + D\Delta n \tag{1.3}$$

where n is the size of the population per unit volume and $m(n)$ is the mass of food [2]. If it is assumed that the mass of food varies as $m(n) = m_0(1 - n/n_0)$, then Eq. (1.3) is a Burgers–Huxley equation with $\alpha = 0$.

If $\alpha = \delta = 0$, Eq. (1.1) is the Kolmogorov–Petrovskii–Piskunov equation $U_t = DU_{xx} + f(x, t)$ c $f(x, t) = \beta U + \gamma U^2$, which was investigated in [3].

Using the linear change of variables $U = aU'$, $x = bx'$ and $t = ct'$, and properly choosing a, b, and c, one can fix any three coefficients (preserving, however, the value of the expressions $D\delta/\alpha^2$ and $\beta\delta/\gamma^2$ during these transformations).

It has been established that Eq. (1.1) does not pass the Painlevé test [4, 5], and consequently it is not an exactly solvable equation. However, one can try to find a certain set of special solutions, and that is the aim of this paper.

Usually, exact solutions of non-linear partial differential equations are found in travelling-wave variables [6], that is, one actually replaces the original equation by an ordinary differential equation

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whose solution is often easy to find. These methods were discussed in detail in [7]. Below it will be shown that other methods yield more general classes of exact solutions than those obtained by transforming to travelling-wave variables.

2. DETERMINATION OF EXACT SOLUTIONS

To find exact solutions of Eq. (1.1), we will use the Cole–Hopf transformation [8, 9]

$$U = AZ_x/Z, \quad Z = Z(x,t) \tag{2.1}$$

We then obtain the equation

$$((\alpha A + 3D)Z_{xx} - Z_t - \gamma A Z_x)Z_x Z + (Z_{xt} - DZ_{xxx} - \beta Z_x)Z^2 + (\delta A^2 - \alpha A - 2D)Z_x^3 = 0 \tag{2.2}$$

Let $Z_x \neq 0$ (otherwise, one obtains the trivial solution $U \equiv 0$, which need not be considered). Equating the coefficients of the different powers of Z to zero, one obtains an over-determined system of equations for $Z(x, t)$

$$\begin{aligned} &Z_{xt} - DZ_{xxx} - \beta Z_x = 0 \\ &(\alpha A + 3D)Z_{xx} - Z_t - \gamma A Z_x = 0 \\ &\delta A^2 - \alpha A - 2D = 0 \end{aligned} \tag{2.3}$$

Note that if $\beta = \gamma = \delta = 0$, Eq. (1.1) becomes the Burgers equation

$$U_t + \alpha U U_x = DU_{xx} \tag{2.4}$$

Then system (2.3) becomes

$$Z_t = DZ_{xx}, \quad \alpha A + 2D = 0 \tag{2.5}$$

This is the well-known Cole–Hopf result: from any solution of the diffusion equation $Z_t - DZ_{xx} = 0$, one can obtain a solution of the Burgers equation (2.4) [10, 11] by using transformation (2.1), where A is determined from the second equation of system (2.5).

In what follows we shall consider the case $\delta \neq 0$.

The last equation of system (2.3) yields

$$A_{1,2} = \frac{\alpha}{2\delta} \pm \sqrt{\frac{\alpha^2}{4\delta^2} + \frac{2D}{\delta}} \tag{2.6}$$

Henceforth it will be assumed that the roots $A_{1,2}$ are real, that is, the expression under the radical on the right of Eq. (2.6) is non-negative.

Substituting the expression for Z_t obtained from the first equation of system (2.3) into the second, we obtain the system

$$Z_t = DZ_{xx} + \beta Z + C_1(t) \tag{2.7}$$

$$(2D + \alpha A)Z_{xx} - \gamma A Z_x - \beta Z - C_1(t) = 0 \tag{2.8}$$

where $C_1(t)$ is a function of t.

The function $Z(x, t)$, determined by system (2.7), (2.8), is reduced by the Cole–Hopf formula (2.1) to a solution of the original equation (1.1). We shall investigate its behaviour as a function of the parameters of the original equation.

3. INVESTIGATION OF THE DEPENDENCE OF THE EXACT SOLUTIONS ON THE PARAMETERS OF THE EQUATION

The form of the solution of system (2.7), (2.8) depends on the parameters of the equation β and γ. The solution of Eq. (2.8) also depends on two functions of t, defined by substitution into Eq. (2.7).

The case $\beta = 0, \gamma = 0$. Solving Eq. (2.8), we have

$$Z(x, t) = C_1(t)\frac{x^2}{2(2D + \alpha A)} + C_2(t) + C_3(t)x \quad (3.1)$$

Substituting the expression obtained into Eq. (7), we find that

$$C_1(t) = c_1, \quad C_2(t) = \left(1 + \frac{D}{2D + \alpha A}\right)c_1 t + c_2, \quad C_3(t) = c_3$$

where c_1, c_2 and c_3 are arbitrary constants. Substituting these relations into Eq. (3.1) and redefining c_1, we obtain

$$Z(x, t) = c_1 x^2/2 + c_2 + c_3 x + (\alpha A + 3D)c_1 t$$

Using the Cole–Hopf transformation (2.1), we find

$$U = A\frac{c_1 x + c_3}{c_1 x^2/2 + c_2 + c_3 x + (\alpha A + 3D)c_1 t} \quad (3.2)$$

If $c_1 = 0$, we have

$$U = A\frac{c_3}{c_3 x + c_2} \quad (3.3)$$

The function (3.3) becomes infinite at the point $x = -c_2/c_3$ and does not depend on time.

If $c_1 \neq 0$, then, dividing the numerator and denominator of expression (3.2) by c_1 and redefining the constants c_2 and c_3, we have

$$U = A\frac{x + c_3}{x^2/2 + c_3 x + c_2 + (\alpha A + 3D)t} \quad (3.4)$$

The behaviour of the function (3.4) is determined by the initial conditions and by the sum $\alpha A + 3D$. The form of the function at the of time may vary: (a) If $c_3^2 - 2c_2 > 0$, the function (3.4) at $t = 0$ becomes infinite at two points: $x_{1,2} = c_1 \pm \sqrt{c_1^2 - 2c_2}$, (b) if $c_3^2 - 2c_2 = 0$, the function has one singular point $x = c_1$, (c) if $c_3^2 - 2c_2 < 0$, there are no singular points. The dependence of the function (3.4) on time is defined by the value of the expression $\alpha A + 3D$. When $\alpha A + 3D > 0$, the amplitude of the solution decreases with time. If $\alpha A + 3D < 0$, the function shows the behaviour of an evolution process with peaking and the amplitude of function (3.4) increases with time. If $\alpha A + 3D = 0$, solution (3.4) does not vary with time.

The behaviour of solution (3.4) as a function of x and t when $\alpha A + 3D > 0$ and $c_3^2 - 2c_2 > 0$ is illustrated in Fig. 1.

The solutions just found belong to the class of functions which are bounded on the real axis for $c_3^2 - 2(c_2 + (\alpha A + 3D)t) < 0$.

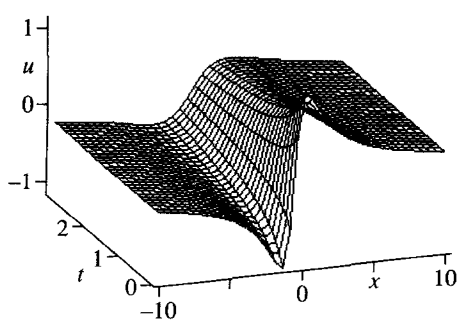

Fig. 1

The case $\beta = 0, \gamma \neq 0$. Equation (2.8) yields

$$Z(x,t) = -\frac{C_1(t)}{\gamma A}x + C_2(t) + C_3(t)\exp(\lambda x) \tag{3.5}$$

where we have put

$$\lambda = \frac{\gamma A}{2D + \alpha A} = \frac{\gamma}{\delta A} \tag{3.6}$$

Using Eq. (2.7), we find the functions $C_1(t)$, $C_2(t)$ and $C_3(t)$ as

$$C_1(t) = c_1, \quad C_2(t) = c_1 t + c_2, \quad C_3(t) = c_3\exp(D\lambda^2 t)$$

where c_1, c_2 and c_3 are arbitrary constants. We substitute these relations into Eq. (3.5). Using the explicit form of λ in (3.6) and redefining the constants c_1 and c_2, we apply the Cole–Hopf transformation (2.1) and obtain

$$U = \frac{\gamma}{\delta}\frac{c_1 - c_3\exp(\lambda x + D\lambda^2 t)}{c_1\lambda x + c_2 - c_3\exp(\lambda x + D\lambda^2 t) - c_1\gamma^2 t/\delta} \tag{3.7}$$

Changes in λ correspond to expansion, contraction or inversion of the graphs; we will therefore consider the case when $\lambda = 1$.

If $c_3 = 0$, then $c_1 \neq 0$ (otherwise $U \equiv 0$). Then, redefining the constant c_2, we deduce from (3.7) that

$$U = \frac{A}{x + c_2 - \gamma^2 t/\delta} \tag{3.8}$$

The function (3.8) has one singular point, $x = \gamma^2 t/\delta - c_2$.

If $c_3 \neq 0$, then dividing it into the numerator and denominator of expression (3.7) and renaming the constants c_1 and c_2, we obtain

$$U = \frac{\gamma}{\delta}\frac{c_1 - \exp(x + Dt)}{c_1(x + Dt) + c_2 - \exp(x + Dt) - c_1 t(D + \gamma^2/\delta)} \tag{3.9}$$

The denominator of this function will have a different number of zeros at $t = 0$, depending on c_1 and c_2. Put $c_* = c_1 - c_1\ln c_1$. Then the function (3.9) will go to infinity twice at the initial instant of time if $\{c_1 > 0, c_2 > c_*\}$ and will have a single singular point if $\{c_1 > 0, c_2 = c_*\}$, $\{c_1 = 0, c_2 > 0\}$ or $c_1 < 0$. The function (3.9) will be bounded at $t = 0$ if either of the conditions $\{c_1 = 0, c_2 \leqslant 0\}$ or $\{c_1 > 0, c_2 < c_*\}$ is satisfied.

The evolution with time of solution (3.9) may be regarded as simultaneous translation along the x axis and variation of the parameter c_2, the size of the translation corresponding to the parameter D and the variation of c_2 being $(\gamma^2/\delta + D)c$. If $(\gamma^2/\delta + D)c_1 = 0$, the solution (3.9) is expressed in terms of travelling-wave variables. In the case when $c_1 > 0$, $c_2 < c_*$ and $\gamma^2/\delta + D > 0$ the function (3.9) decays with time; its behaviour is similar to that shown in Fig. 2. But if $\gamma^2/\delta + D < 0$, a bounded solution will exist only for a finite time.

The solutions just found belong to the class of functions bounded on the real axis provided $\{c_1 > 0, c_2 - c_1(\gamma^2/\delta + D)t < c_1(1 - \ln c_1)\}$ or $\{c_1 = 0, c_2 \leqslant 0\}$.

The case $\beta \neq 0, \gamma^2 + 4\delta\beta = 0$. Equation (2.8) yields

$$Z(x,t) = -C_1(t)/\beta + (C_2(t) + C_3(t)x)\exp(\lambda x); \quad \lambda = \gamma/(2\delta A) \tag{3.10}$$

Substituting expression (3.10) into Eq. (2.7), we have

$$C_1(t) = c_1, \quad C_2(t) = (c_2 + 2\lambda D c_3 t)\exp((D\lambda^2 + \beta)t), \quad C_3(t) = c_3\exp((D\lambda^2 + \beta)t)$$

where c_1, c_2 and c_3 are arbitrary constant. Hence we obtain

$$Z(x,t) = -c_1/\beta + (c_2 + 2\lambda D c_3 t + c_3 x)\exp(\lambda x + (D\lambda^2 + \beta)t)$$

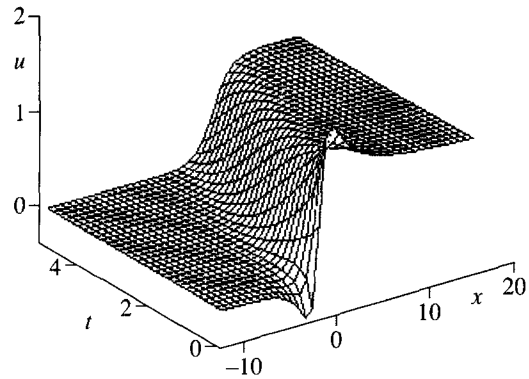

Fig. 2

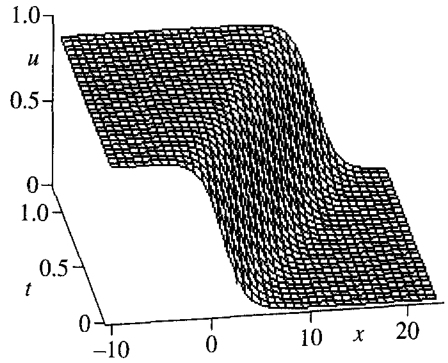

Fig. 3

Then, by the Cole–Hopf formula (2.1), we find that

$$U = A\frac{\lambda(c_2 + 2\lambda D c_3 t) + \lambda c_3 x + c_3}{c_2 + 2\lambda D c_3 t + c_3 x - (c_1/\beta)\exp(-\lambda x - (D\lambda^2 + \beta)t)} \tag{3.11}$$

If $c_3 = 0$, then, redefining the constant c_1, we obtain a solution in travelling-wave variables

$$U = \frac{\lambda A}{c_1 \exp(-\lambda x - (D\lambda^2 + \beta)t) + 1} \tag{3.12}$$

The function (3.12) has one singular point $x = (\ln(-c_1) - (D\lambda^2 + \beta)t)/\lambda$ when $c_1 < 0$. If $c_1 \geqslant 0$, the solution is bounded over the entire real axis (this case is illustrated in Fig. 3).

If $c_3 \neq 0$, then, dividing the numerator and denominator of expression (3.11) by c_3 and redefining the constants c_1 and c_2, we obtain

$$U = A\frac{\lambda(x + c_2 + 2D\lambda t) + 1}{x + c_2 + 2D\lambda t + c_1\exp(-\lambda(x + c_2 + 2D\lambda t) + (D\lambda^2 - \beta)t)} \tag{3.13}$$

After the notation $S_1 = c_1\exp((D\lambda^2 - \beta)t)$ and $S_2 = c_2 + 2D\lambda t$ is introduced, formula (3.13) becomes

$$U = A\frac{\lambda(x + S_2) + 1}{(x + S_2) + S_1\exp(-\lambda(x + S_2))} \tag{3.14}$$

Whether the function (3.14) is bounded or not depends on the product $S_1\lambda$. If $S_1\lambda \leqslant 0$ or $S_1\lambda e = 1$ (where e is the base of natural logarithms), the function (3.14) will have a single singular point. If $0 < S_1\lambda e < 1$, the solution goes to infinity twice. If $S_1\lambda e > 1$, the function (3.14) is bounded over the entire real axis (this is the situation illustrated in Fig. 2). Dependence on time is equivalent to the variation of S_1 and S_2, while the sign of the expression $S_1\lambda$ does not change.

The case $\beta \neq 0, \gamma^2 + 4\delta\beta > 0$. Put

$$\lambda_{1,2} = \frac{\gamma \pm \sqrt{\gamma^2 + 4\delta\beta}}{2\delta A}$$

with the sign chosen in such a way that $|\lambda_1| > |\lambda_2|$. It then follows from Eq. (2.8) that

$$Z(x, t) = -C_1(t)/\beta + C_2(t)\exp(\lambda_1 x) + C_3(t)\exp(\lambda_2 x) \tag{3.15}$$

The functions $C_1(t)$, $C_2(t)$, $C_3(t)$ are found from Eq. (2.7):

$$C_1(t) = c_1, \quad C_2(t) = c_2\exp((D\lambda_1^2 + \beta)t), \quad C_3(t) = c_3\exp((D\lambda_2^2 + \beta)t)$$

where c_1, c_2, and c_3 are arbitrary constants. Then, substituting these functions into expression (3.15), we obtain

$$Z(x, t) = c_1 + c_2E_1(x, t) + c_3E_2(x, t); \; E_i(x, t) = \exp(\lambda_i x + (D\lambda_i^2 + \beta)t), \quad i = 1, 2$$

Next, using formula (2.1), we calculate

$$U = A\frac{\lambda_1 c_2 E_1 + \lambda_2 c_3 E_2}{c_1 + c_2 E_1 + c_3 E_2} \tag{3.16}$$

If $c_2 = 0$, then $c_3 \neq 0$ (otherwise $U \equiv 0$). Then, redefining the constant c_1, we have

$$U = A\frac{\lambda_2 E_2}{c_1 + E_2} = \frac{A\lambda_2}{1 + c_1\exp(-\lambda_2 x - (D\lambda_2^2 + \beta)t)} \tag{3.17}$$

At $t = 0$ the function $U(x, t)$ becomes infinite at the point $x = -\ln(-c_1)$, provided that $c_1 < 0$. If $c_1 \geqslant 0$, then the solution (3.17) is bounded (it is illustrated in Fig. 3). This is a function of travelling-wave variables, the passage of time being equivalent to translation along the x axis.

If $c_2 \neq 0$, then, dividing into it the numerator and denominator of expression (3.16) and redefining the constants c_1 and c_3, we have

$$U = A\frac{\lambda_1 E_1 + \lambda_2 c_3 E_2}{c_1 + E_1 + c_3 E_2} \tag{3.18}$$

Variation of λ_1 is equivalent to variation of the scales of the axes. We introduce a new variable $y = \lambda_1 x$ and put $p = \lambda_2/\lambda_1(|p| < 1)$. Then the function (3.18) has the form

$$U = A\lambda_1\frac{\exp(y) + S_1 p\exp(py)}{S_2 + \exp(y) + S_1\exp(py)} \tag{3.19}$$

where the parameters S_1 and S_2 depend on time and vary as given by

$$S_1 = c_3\exp(D(\lambda_2^2 - \lambda_1^2)t), \quad S_2 = c_1\exp(-(D\lambda_1^2 + \beta)t)$$

Depending on the parameters S_1, S_2 and p, the denominator of the function (3.19) will vanish at a different number of points (the function itself will have poles of the first order at these points). There are three possible versions: (a) two zeros, if $\{p < 0, S_1 > 0, S_2 < S_*\}$ or $\{p > 0, S_1 < 0, 0 < S_2 < S_*\}$, (b) one zero, if $\{S_1 p < 0, S_2 = S_*\}$, $\{p > 0, S_1 > 0, S_2 < 0\}$, $\{p > 0, S_1 < 0, S_2 \leqslant 0\}$, $\{p < 0, S_1 < 0\}$ or $\{S_1 = 0, S_2 < 0\}$, (c) no zeros if $\{S_1 = 0, S_2 \geqslant 0\}$ (the solution is expressed in terms of travelling-wave variables; it is illustrated in Fig. 3), $\{S_1 p < 0, S_2 > S_*\}$ (in that case the behaviour of the solution is similar to that illustrated in Fig. 2), or $\{p > 0, S_1 > 0, S_2 \geqslant 0\}$ (this situation is demonstrated in

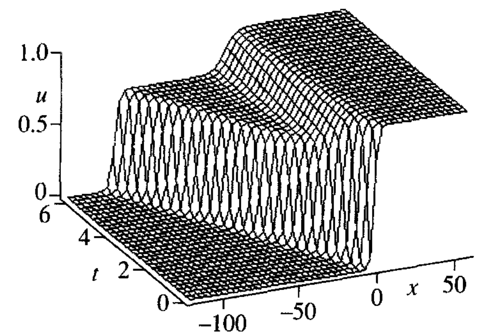

Fig. 4

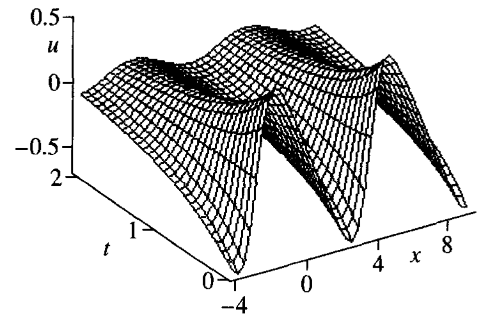

Fig. 5

Fig. 4). We have put $S_* = ((1-p)/p)(-S_1 p)^{1/(1-p)}$ here. The passage of time is equivalent to variation of S_1 and S_2.

The case $\gamma^2 + 4\beta\delta < 0$. Equation (2.8) yields

$$Z(x, t) = C_1(t) + \exp(kx)(C_2(t)\sin(lx) + C_3(t)\cos(lx))$$

where

$$k = \frac{\gamma}{2\delta A}, \quad l = \frac{\sqrt{-(\gamma^2 + 4\beta\delta)}}{2\delta A}$$

Determining the functions $C_1(t)$, $C_2(t)$, $C_3(t)$ from Eq. (2.7), we obtain

$$Z(x, t) = c_1 + c_2 F_1(x, t) + c_3 F_2(x, t)$$

where

$$F_1(x, t) = E(x, t)\sin(lx + 2Dklt), \quad F_2(x, t) = E(x, t)\cos(lx + 2Dklt)$$

$$E(x, t) = \exp(kx + (D(k^2 - l^2) + \beta)t)$$

Here c_1, c_2 and c_3 are arbitrary constants. Then, by the Cole–Hopf formula (2.1),

$$U = A\frac{(kc_2 - lc_3)F_1 + (lc_2 + kc_3)F_2}{c_1 + c_2F_1 + c_3F_2} \tag{3.20}$$

If $\gamma \neq 0$, the function (3.20) will have an infinite number of singular points.

If $\gamma = 0$, then $k = 0$. Put $f = Dl^2 - \beta$. Then

$$U = Al\frac{c_2\cos(lx) - c_3\sin(lx)}{c_1\exp(ft) + c_2\sin(lx) + c_3\cos(lx)} \tag{3.21}$$

It follows from formula (3.21) that $U(x, t)$ is periodic in x with period $2\pi/l$, and the position of its zeros does not depend on time. If $c_1 \neq 0$ and $f > 0$, the solution decays with time and the poles disappear with time. This case is illustrated in Fig. 5.

Note that exact solutions of Eq. (1.1), similar to those described above, may also be obtained using a generalization of transformation (2.1), namely

$$U = AZ_x/Z + B, \quad Z = Z(x, t)$$

The function $Z(x, t)$ and the constants A and B are defined in a manner analogous to that described above.

Thus, exact solutions have been obtained for the Burgers–Huxley equation (1.1) using the Cole–Hopf transformation (2.1). It has been established that if $\beta = \gamma = 0$ the solution is a rational function of x and t. When $\beta = 0, \gamma \neq 0$ and $\beta \neq 0, \gamma^2 + 4\delta\beta \geqslant 0$, the exact solutions of Eq. (1.1) are expressed in terms of exponential functions and polynomials in x and t. If $\gamma = 0$ and $\beta\delta < 0$, a solution exists which is a periodic function of x and decays with time.

REFERENCES

1. OSIPOV, V. V., The simplest autowaves. *Sorosovskii Obrazovatel'nyi Zhurnal*, 1999, 7, 115–121.
2. LOSKUTOV, A. Yu. and MIKHAILOV, A. S., *Introduction to Synergetics*. Nauka, Moscow, 1990.
3. KOLMOGOROV, A. N., PETROVSKII, I. G. and PISKUNOV, N. S., Investigation of the diffusion equation coupled with an increase in the quantity of matter and its application to a biological problem. *Byul. Mosk. Gos. Univ., Ser. A*, 1937, 6, 1–26.
4. KUDRYASHOV, N. A., *Analytical Theory of Non-linear Differential Equations*. Mosk. Inzk–Firz. Inst, Moscow, 2002.
5. KUDRYASHOV, N. A., Partial differential equations with solutions having movable first-order singularities. *Phys. Lett. A.*, 1992, **169**, 4, 237–242.
6. KUDRYASHOV, N. A., Exact soliton solutions of the generalized evolution equation of wave dynamics. *Prikl. Mat. Mekh.*, 1988, **52**, 3, 465–470.
7. KUDRYASHOV, N. A. and SUKHAREV, M. B., Exact solutions of a non-linear fifth-order equation for describing waves on water. *Prikl. Mat. Mekh.*, 2001, **65**, 5, 884–894.
8. COLE, J. D., On a quasi-linear parabolic equation occurring in aerodynamics. *Q. Appl. Math.*, 1951, **9**, 3, 225–236.
9. HOPE, E., The partial differential equation $u_t + uu_x = uu_{xx}$. *Comm. Pure Appl. Math.*, 1950, **3**, 3, 201–230.
10. BURGERS, J. M., A mathematical model illustrating the theory of turbulance. In *Advances in Applied Mathematics*, Vol. 1. Academic Press, New York, 1948, pp. 171–199.
11. ROZHDESTVENSKII, B. L. and YANENKO, N, N., *Systems of Quasilinear Equations and Their Application to Gas Dynamics*. Nauka, Moscow, 1978.

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